DELHI PUBLIC SCHOOL

MATHEMATICS ANSWER KEY

SECTION – A

1. If A = $\{1,2,3,4,5,6\}$, write the relation aRb such that a+ b = 7 , a,b $\in$ A .

ANS R = $\{(1,6),(2,5),(3,4),(5,2),(6,1)\}$ (1m)

2. If A = $\begin{bmatrix}3\\5\\2\end{bmatrix}$ and B = $\begin{bmatrix}1 & 0 & 4\end{bmatrix}$, find $(AB)^T$.

ANS $(AB)^T = \begin{bmatrix}3 & 5 & 2\\0 & 0 & 0\\12 & 20 & 8\end{bmatrix}$ (1m)

3. Find the value of x if $\begin{vmatrix}1 & 2\\4 & 9\end{vmatrix} = \begin{vmatrix}3 & 3\\6x & 4x\end{vmatrix}$.

ANS 9-8=12x-18x (1/2 m)

-1=-6x

x= -1/6 (1/2 m)

4. Find the value of $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$.

ANS $\cos\theta = \cos(-\sqrt{3}/2) = \cos 5\pi/6$ (1/2 m)

$=\cos\left(\frac{5\pi}{6}+\frac{\pi}{6}\right) = \cos\pi = -1$ (1/2 m)

5. Evaluate $\int_0^{1.5}[x]$ dx where $[x]$ is greatest integer function .

ANS $\int_0^{1.5}[x]\,dx=\int_0^{1}[x]\,dx+\int_1^{1.5}[x]\,dx$ (1/2 m)

$= 0 + (x)_1^{1.5} = 1.5 – 1 = 0.5$ (1/2 m)

6. If $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$, $\vec{c} = \hat{k} + \hat{i}$, find a unit vector in the direction of $\vec{a} + \vec{b} + \vec{c}$.

ANS $\vec{a} + \vec{b} + \vec{c} = 2(\hat{i} + \hat{k} + \hat{j})$ (1/2 m)

Therefore Unit vector in the direction of

$$\vec{a} + \vec{b} + \vec{c} = \frac{2(\hat{i} + \hat{j} + \hat{k})}{2\sqrt{3}} = \pm \frac{(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3}}$$ (1/2 m)

7. Find the projection of the vector $\vec{a} = 2\hat{i} - 5\hat{j} + 3\hat{k}$ on the vector $\vec{b} = 4\hat{i} - 5\hat{j} + 3\hat{k}$.

ANS The projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}$ = $\vec{a}.\vec{b}$ = 8+25+9= 42 (1/2 m)

= $|\vec{b}|^2$ = $\sqrt{50}$

Therefore The projection of $\vec{a}$ on $\vec{b}$ = 42/50 = 21/25 . (1/2 m)

8. If sum of two vectors is a unit vector ,prove that the magnitude of their difference is $\sqrt{3}$.

ANS Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be unit vectors

$$(\hat{a} + \hat{b})^2 = (\hat{c})^2$$

$$|\hat{a}|^2 + |\hat{b}|^2 = |\hat{c}|^2$$

$2\hat{a}\,\hat{b}$ = -1 (1/2 m)

$(\hat{a} - \hat{b})^2 = (\hat{a})^2 + (\hat{b})^2 - 2\,\hat{a}\,\hat{b}$

$$\left|\hat{a} - \hat{b}\right|^2 = (\hat{a})^2 + (\hat{b})^2 - 2\,\hat{a}\,\hat{b} = 3$$

$$\left|\hat{a} - \hat{b}\right| = \sqrt{3}$$ (1/2 m)

9. If A is a square matrix of order 3 such that $|adjA| = 64$, find $|A|$.

ANS $|AdjA| = |A|^{3-1} = |A|^2$ (1/2 m)

$|A| = \pm 8$ (1/2 m)

10. Find the point on the curve y = x^2 -2x + 3 , where the tangent is parallel to x-axis .
ANS Since the tangent is parallel to x- axis dy/dx = 0
2x-2 = 0 (1/2 m)

x =1
y = 2 therefore point is (1,2) (1/2 m)

SECTION – B

11. Prove that the relation R in the set A = $\{1,2,3,4,5\}$ given by
R = $\{(a,b) : |a-b| \text{is even}\}$,is an equivalence relation .

Ans Given set A = $\{1,2,3,4,5\}$ and relation R = $\{(a,b) : |a-b| \text{is even}\}$, we have to show that it is equivalence relation . For equivalence relation , it must be reflexive , symmetric and transitive .

i) Reflexive : aRa $\Rightarrow$ $|a-a| = 0$ (even) is true .
Hence (a,a) $\in$ R . So it is reflexive for all elements of set A . (1 m)

ii) Symmetric : Let (a,b) $\in$ R $\Rightarrow |a-b|$ is an even number

$\Rightarrow |b-a|$ is also an even number

$\Rightarrow$ (b , a) $\in$ R

Hence R is symmetric relation . (1 m)

iii) Transitive : If (a,b) $\in$ and (b,c) $\in$ R $\Rightarrow |a-b|$ is an even number and

$|b-c|$ is also an even number .Now , let us consider

$$|a-c| = |(a-b)+(b-c)| = |even + even|$$

$\Rightarrow$ $|a-c|$ is an even number

$\Rightarrow$ (a,c) $\in$ R

So relation is transitive (1 m)

Since it satisfy all three conditions of equivalence relation ,so R is an equivalence relation(1 m)

12. Solve the following equation:

$$3\sin^{-1}\left(\frac{2x}{1+x^2}\right) - 4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3} .$$

Ans $3\sin^{-1}\left(\frac{2x}{1+x^2}\right) - 4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$

$3 \times 2\tan^{-1}x - 4 \times 2\tan^{-1}x \times 2\tan^{-1}x = \frac{\pi}{3}$ (1 m)

$6\tan^{-1}x - 8\tan^{-1}x + 4\tan^{-1}x = \frac{\pi}{3}$

$2\tan^{-1}x = \frac{\pi}{3}$ (1 m)

$\tan^{-1}x = \frac{\pi}{6}$ (1 m)

$x = \frac{1}{\sqrt{3}}$. (1 m)

OR

Find the value of $2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right)$.

Ans . $2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right)$

$= 2\left[\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)\right] + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$ (1 m)

$\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$= 2\left[\tan^{-1}\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right] + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$ (1 m)

$= 2\tan^{-1}\frac{13}{39} + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$

$= 2\tan^{-1}\frac{1}{3} + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$

$$=\tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) \quad \left(2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right)$$

$$=\tan^{-1}\left(\frac{3}{4}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$$

Now $\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)=x$ (1 m)

Therefore tanx = 1/7

Therefore x = $\tan^{-1}\left(\frac{1}{7}\right)$

Therefore $\tan^{-1}\left(\frac{3}{4}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)=\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

$$=\tan^{-1}\left(\frac{\frac{21+4}{28}}{\frac{25}{28}}\right)=\tan^{-1}(1)=\frac{\pi}{4}.$$ (1 m)

13 Using properties of determinants , prove that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & a+c+2b \end{vmatrix} = 2(a+b+c)^3 .$$

Ans13 . L.H.S.= $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & a+c+2b \end{vmatrix}$

On applying $C_1 \to C_1 + C_2 + C_3$ we get

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & a+c+2b \end{vmatrix} \quad \text{(1 m)}$$

Taking out $2(a+b+c)$ common from C_1

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & a+c+2b \end{vmatrix} \quad \text{(1 m)}$$

= $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+2a & 0 \\ 0 & 0 & a+c+b \end{vmatrix} \quad \text{(1 m)}$$

Expanding along C_1

$$= 2(a+b+c) \begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix}$$

$= 2(a+b+c)(a+b+c)^2 = 2(a+b+c)^3$ = R.H.S. (1 m)

14. A man takes a step forward with probability 0.4 and backwards with probability 0.6 . Find the probability that at the end of the eleven steps he is one step away from the starting point .

Ans14. Suppose taking a step forward and backwards be regarded as a success and failure , respectively ,then in our problem (of Bernoullian trials)

$P = 0.4 = \frac{2}{5}$, q = 0.6 = 3/5 and n = 11 (1 m)

The man will be one step away from the starting point if he takes .

i) 6 steps forward and 5 steps backwards or

ii) 5 steps forward and 6 steps backwards

Hence , the required probability = P(6) +P(5) (1 m)

$$= {}^{11}C_6 \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right)^6 + {}^{11}C_5 \left(\frac{3}{5}\right)^6 \left(\frac{2}{5}\right)^5 \quad \text{(1 m)}$$

$$= {}^{11}C_5 \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right)^5 \text{ Ans} \quad \text{(1 m)}$$

15. Determine the values of a,b and c for which the function

$$f(x)=\begin{cases} \dfrac{\sin(a+1)x+\sin x}{x}, x<0 \\ c, x=0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{b\sqrt{x^3}}, x>0 \end{cases}$$ is continuous at x= 0 .

Ans . Here f(0) = c

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(a+1)x+\sin x}{x} = \lim_{x\to 0^-} \left(\frac{\sin(a+1)x}{x}+\frac{\sin x}{x}\right)$$

$$= \lim_{x\to 0^-} \frac{\sin(a+1)x}{x} \text{ x } (a+1) + \lim_{x\to 0^-} \frac{\sin x}{x}$$

$$= (a+1) = \lim_{x\to 0^-} \frac{\sin(a+1)x}{x} + 1$$

as $x \to 0^-$ then $(a+1)x \to 0^-$

= a+1 +1 = a+2

$\text{Limf(x)} = \lim\limits_{x \to 0^+} \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{b\sqrt{x^3}}$

$\text{Lim} = \lim\limits_{x \to 0^+} \dfrac{\sqrt{x}(\sqrt{1+bx}-1)}{bx\sqrt{x}} \quad = \lim\limits_{x \to 0^+} \dfrac{(\sqrt{1+bx}-1)}{bx} \times \left(\dfrac{\sqrt{1+bx}+1}{\sqrt{1+bx}+1}\right)$

$\lim\limits_{x \to 0^+} \dfrac{1+bx-1}{bx(\sqrt{1+bx}+1)}$

$= \lim\limits_{x \to 0^+} \dfrac{1}{\sqrt{1+bx}+1} \quad = \dfrac{1}{2}$

Now , f is continuous at x= 0 if $\lim\limits_{x \to 0^-} f(x) = f(0) = \lim\limits_{x \to 0^+} f(x)$

i.e. a+2 = c ½

therefore a= -3/2 and c= ½ .
Hence for f to be continuous at 0 we must have a= -3/2 and c= ½ , b may be any real value .

16. Evaluate $\displaystyle\int \frac{1}{1+\tan^4 x}\, dx$

Ans substitute tanx = t
$\therefore$ $\sec^2 x\, dx = dt$ (1 m)

$$\int \frac{1}{1+\tan^4 x}\, dx = \int \frac{dt}{(1+t^4)(1+t^2)} = \frac{1}{2}\int \frac{(t^4+1)-(t^4-1)}{(1+t^4)(1+t^2)}dt \qquad \text{(1 m)}$$

$$= \frac{1}{2}\int \frac{1}{1+t^2}dt - \frac{1}{2}\int \frac{t^4-1}{(1+t^4)(1+t^2)}dt \qquad \text{(1 m)}$$

$$= \frac{1}{2}\tan^{-1} t - \frac{1}{2}\int \frac{t^2-1}{t^4+1}dt = \frac{1}{2}\tan^{-1} t \text{ -1/2 } I_1$$

Where $I_1 = \int \frac{t^2-1}{t^4+1}dt$ divide N and D by t^2

$$= \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt \quad \text{put } t+1/t = u \Rightarrow (1-1/t^2)dt = du \qquad \text{(1 m)}$$

$$\int \frac{du}{u^2-2} = \frac{1}{\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|$$

$$= \frac{1}{\sqrt{2}}\log\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right| = \frac{1}{\sqrt{2}}\log\left|\frac{t^2-\sqrt{2}t+1}{t^2+\sqrt{2}t+1}\right| \quad \text{ANS} \qquad \text{(1 m)}$$

17. Show that the curves $y^2 = 8x$ and $2x^2 + y^2 = 10$ intersects orthogonally at the point $(1, 2\sqrt{2})$.

Ans Given curves are : $y^2 = 8x$ and
$2x^2 + y^2 = 10$
Diff. (i) and (ii) w.r.t x we get

$$2y\frac{dy}{dx} = 8$$

$$\frac{dy}{dx} = \frac{4}{y} \qquad \text{(1 m)}$$

The slope of (i) curve at the given point (1, $2\sqrt{2}$) is $m_1 = \left(\frac{dy}{dx}\right)_{(1,\,2\sqrt{2})} = \frac{4}{2\sqrt{2}} = \sqrt{2}$

And from (ii) curve $\frac{dy}{dx} = \frac{-2x}{y}$

Therefore the slope of (ii) curve at point (1, $2\sqrt{2}$) $m_2 = \left(\frac{dy}{dx}\right)_{(1,\,2\sqrt{2})} = \frac{-1}{\sqrt{2}}$

(1 m)

Thus $m_1 = \sqrt{2}$ and $m_2 = \frac{-1}{\sqrt{2}}$

Thus $m_1 \times m_2 = \sqrt{2} \times \frac{-1}{\sqrt{2}} = -1$ (1 m)

Since the product of slopes of both the curves at point (1, $2\sqrt{2}$) is equal to -1 . Hence the curve cuts orthogonally or perpendicularly at point (1, $2\sqrt{2}$). (1 m)

OR

An airforce plane is ascending vertically at the rate of 100 km/hr . If the radius of earth is r km , how fast is the area of the Earth ,visible from the plane increasing at 3 minutes after it started ascending ? It is given that if h is the height of the plane above the Earth, then visible area is equal to $\frac{2\pi r^2 h}{r+h}$.

Ans . Let h and A be respectively height of the plane above the earth and visible area from the plane at time t .

$A = \frac{2\pi r^2 h}{r+h}$ (1/2 m)

The height of plane is increasing at the rate of 100 km/hr

$\frac{dh}{dt} = 100$, rate of change of visible area A w.r.t to time

$$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt} \quad = 2\pi r^2 \frac{d}{dh}\left(\frac{h}{r+h}\right) \times 100$$ (1 m)

$$= \frac{200\pi r^2}{(r+h)^2}$$

Height of plane after 3 min. = $\frac{3}{60} \times 100$ = 5 km therefore h = 5 km. (1 m)

Rate of increase of visible area w.r.t time after 3 min. $\frac{200\pi r^2}{(r+h)^2}$ sq.km/hr. (1 m)

18. If x = a$\left[\cos t + \log\left|\tan\frac{t}{2}\right|\right]$ and y = a sin t ; then find $\frac{dy}{dx}$ at t = $\frac{\pi}{3}$.

ANS 18. We have x = a$\left[\cos t + \log\left|\tan\frac{t}{2}\right|\right]$

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan\frac{t}{2}} \frac{d}{dt}\left(\tan\frac{t}{2}\right)\right] = a\left[-\sin t + \frac{\sec^2 t}{\tan\frac{t}{2}} \frac{d}{dt}\left(\frac{t}{2}\right)\right]$$ (1 m)

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{2\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}\cos^2\frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$= a\frac{\cos^2 t}{\sin t}$ (1)

y= a sint

$\Rightarrow \frac{dy}{dt} = a\cos t$ (2) (1 m)

Dividing (2) by (1)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{a\cos^2 t/\sin t}$$

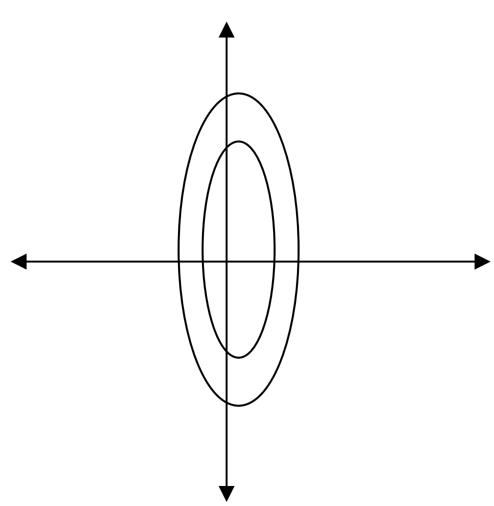
$$\frac{dy}{dx} = \tan t$$ (1 m)

$$\left(\frac{dy}{dx}\right)_{t=\pi/3} = \tan\frac{\pi}{3} = \sqrt{3} \text{ ANS}$$ (1 m)

19. Obtain the differential equation of ellipse having foci on y – axis and centre at the origin .

Ans.19. We know that equation of the family of ellipse is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (1/2 m)

FIGURE



Diff.w.r.t x

$$\frac{2x}{b^2}+\frac{2y}{a^2}\frac{dy}{dx}=0$$

$$\frac{y}{x}\left(\frac{dy}{dx}\right)=-\frac{a^2}{b^2}$$

(1 m)

Again diff. w.r.t x

$$\frac{y}{x}\left(\frac{d^2y}{dx^2}\right)+\left[\frac{x\frac{dy}{dx}-y}{x^2}\right]\frac{dy}{dx}=0$$ (1 m)

$$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$ which is the required answer . (1 m)

20.Evaluate $\int_0^{\pi}\frac{x}{a^2\cos^2 x+b^2\sin^2 x}$ dx

Ans $\int_0^{\pi}\frac{x}{a^2\cos^2 x+b^2\sin^2 x}$ dx = $\int_0^{\pi}\frac{(\pi-x)}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}$ dx

Using $\int_0^{a} f(x)dx=\int_0^{a} f(a-x)dx$ (1 m)

$$=\pi\int_0^{\pi}\frac{dx}{a^2\cos^2 x+b^2\sin^2 x}-\int_0^{\pi}\frac{xdx}{a^2\cos^2 x+b^2\sin^2 x}$$

$$= \pi \int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x} \text{ -I}$$ (1 m)

$$2I = \pi \int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$$

$$I = \frac{\pi}{2} \int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x} = \pi . \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x}$$

$$= \pi \int_0^{\pi} \frac{\sec^2 x dx}{a^2 + b^2 \tan^2 x}$$ on dividing by $\cos^2$x . (1 m)

Now put b tanx = t . On differentiating $\sec^2$xdx = dt and x$\rightarrow 0, t \rightarrow \pi/2, t \rightarrow \infty$

$$I = \frac{\pi}{b} \int_0^{\infty} \frac{dt}{a^2 + t^2} = \left[\frac{\pi}{ab} \tan^{-1} \frac{t}{a} \right]_0^{\infty} = \frac{\pi^2}{2ab}.$$

$$\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x} \mathbf{dx} = \frac{\pi^2}{2ab}$$ **Ans**

21. $\vec{a}, \vec{b}, \vec{c}, \vec{d},$ are the vector forming the consecutive sides of a quadrilateral . Show that a necessary and sufficient condition that the figure be a parallelogram is that $\vec{a} + \vec{c} = \vec{0}$ and that implies $\vec{b} + \vec{d} = \vec{0}$.

ANS .21 Let ABCD be the quadrilateral whose sides AB ,BC ,CD and DA represent the Vectors a,b,c, and d respectively .

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \overrightarrow{a + b}$

$\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

$(\vec{a} + \vec{b}) + \vec{c} = \vec{d}$

$(\vec{a} + \vec{b}) + \vec{c} + \vec{d} = 0$ (1) (1m)

The condition is necessary .

ABCD is a parallelogram if AB = DC and AB is parallel to DC

If $\overrightarrow{AB} = \overrightarrow{DC}$ if $\overrightarrow{AB}$ = -$\overrightarrow{CD}$

If $\vec{a} = -\vec{c}$ if $\vec{a} + \vec{c}$ = 0 , which is true (1m)

Also from (1) $\vec{b} + \vec{d} + \vec{0} = \vec{0}$

The condition is sufficient $\vec{a} + \vec{c} = \vec{0}$ i.e. $\vec{a} = -\vec{c}$

$\Rightarrow \overrightarrow{AB} = -\overrightarrow{CD}$ (1m)

$\Rightarrow \overrightarrow{AB} = \overrightarrow{DC}$

Now from (1) $\vec{b} + \vec{d} = \vec{0}$ i.e. $\vec{b} = -\vec{d}$ (2)

From (2) and (3) ABCD is parallelogram , which is true . (1m)

OR

Find the vector and Cartesian equation of the plane containing the two lines :

$\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 5\hat{k})$

and $\vec{r} = (3\hat{i} + 3\hat{j} - 7\hat{k}) + \mu(3\hat{i} - 2\hat{j} + 5\hat{k})$.

ANS Given lines first passes through the point having position vectors $\overrightarrow{a_1} = 2\hat{i} + \hat{j} - 3\hat{k}$

$\overrightarrow{a_2} = 3\hat{i} + 3\hat{j} - 7\hat{k}$ respectively and are parallel to the vectors

$\overrightarrow{b_1} = \hat{i} + 2\hat{j} + 5\hat{k}$ and $\overrightarrow{b_2} = 3\hat{i} - 2\hat{j} + 5\hat{k}$ resp. (1m)

The plane containing these two lines passes through points having vectors $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$

And is perpendicular to the vectors $\vec{n} = \overrightarrow{b_1} \times \overrightarrow{b_2}$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 5 \\ 3 & -2 & 5 \end{vmatrix} = 20\hat{i} + 10\hat{j} - 8\hat{k} \quad \text{(1m)}$$

Required equation of plane be $(\vec{r} - \overrightarrow{a_1}).\vec{n} = 0$

$$\vec{r}\,.\,\vec{n} = \overrightarrow{a_1}\,.\,\vec{n}$$

$$\vec{r}\,.\,(20\hat{i} + 10\hat{j} - 8\hat{k}) = (2\hat{i} + \hat{j} - 3\hat{k}).(20\hat{i} + 10\hat{j} - 8\hat{k})$$

$$\vec{r}\,.(10\hat{i} + 5\hat{j} - 4\hat{k}) = 37 \quad \text{(1m)}$$

This is the required vector equation of plane .

Put $\vec{r} = x\hat{i} + y\hat{j} - z\hat{k}$ we get

$(x\hat{i} + y\hat{j} - z\hat{k}).(10\hat{i} + 5\hat{j} - 4\hat{k}) = 37$

10x +5y -4z = 37 is the required Cartesian equation of plane . (1m)

22. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane 2x – 3y + 4z -6 = 0 .

ANS 22. Let the coordinate of foot of the perpendicular P drawn from origin is (x_1,y_1,z_1) .Then d,rs of OP are x_1-0 , y_1-0 , z_1-0 i.e. x_1 ,y_1,z_1 .

As point P lies on the plane

Therefore dc's of OP be $\frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$ (1m)

The equation of plane in normal form be $\frac{2}{\sqrt{29}}x - \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{6}{\sqrt{29}}$

Since the d.c's are proportional to dr's

$\frac{x_1}{\frac{2}{\sqrt{29}}} = \frac{y_1}{\frac{-3}{\sqrt{29}}} = \frac{z_1}{\frac{4}{\sqrt{29}}} = k$

$x_1 = \frac{2k}{\sqrt{29}}, y_1 = \frac{-3k}{\sqrt{29}}, z_1 = \frac{4k}{\sqrt{29}}$ (1m)

On substituting these values in the equation of plane we have

k = $\frac{2}{\sqrt{29}}$ (1m)

Therefore co-ordinate of foot of perpendicular is $(\frac{12}{\sqrt{29}}, \frac{-18}{\sqrt{29}}, \frac{24}{\sqrt{29}})$ (1m)

SECTION – C

23. Every gram of wheat provides 0.1 gm of proteins and 0.25 gm of carbohydrates. The corresponding values for rice 0.05 gm and 0.5 gm respectively .Wheat costs Rs.4 per kg and rice Rs. 6 per kg . The minimum daily requirements of proteins and carbohydrates for an average child are 50 gms and 200gms respectively . In what quantities should wheat and rice be mixed in the daily diet to provide minimum daily requirements of proteins and carbohydrates at minimum cost . Frame an L.P.P. and solve it graphically .

Ans 23. The given information is shown in the following table :

	Wheat	Rice	Min. Requirement
Proteins	0.1 gm/gm	0.05/gm	50gm
Carbohydrates	0.25gm/gm	0.5/gm	200gm
Price	Rs 4/kg	Rs 6/kg	

(1m)

Let the quantities of wheat and rice mixed be x gm and y gm resp.

$\therefore x \geq 0, y \geq 0$

Min. requirement of proteins = 50 gm
(0.1)x +0.05 y $\geq$ 50

2x +y $\geq$ 1000
Min. req. of Car. = 200 gm
0.25x +0.5y $\geq$ 200
x + 2y $\geq$ 800 (2m)

Let Z be the total cost

$$Z = 4\left(\frac{x}{1000}\right) + 6\left(\frac{y}{1000}\right) = \frac{x}{250} + \frac{3y}{500}$$

The given LPP reduces to

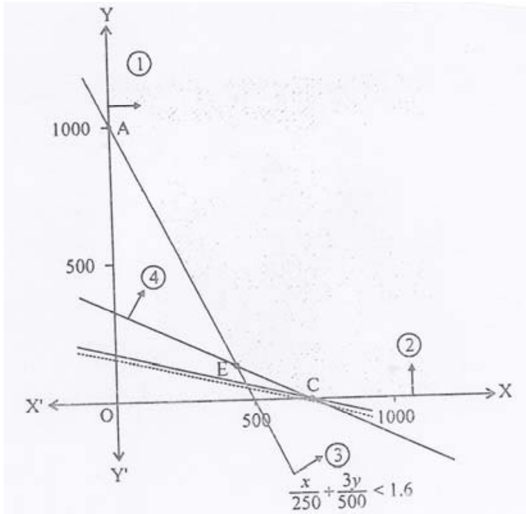
MIN. Z = $\frac{x}{250} + \frac{3y}{500}$ subject to constraints

x$\geq$ 0
y$\geq$ 0
2x +y $\geq$ 1000
x + 2y $\geq$ 800

The vertices of the feasible region are A (0,1000) , E(400,200) and D(800,0)
At A (0,1000) Z= 6
At E (400,200) Z = 2.8
At D(800,0) Z = 3.2 (1m)

Min. (6,2.8 ,3.2) = 2.8
We draw the graph of the inequality is shown in the figure

FIGURE (2m)

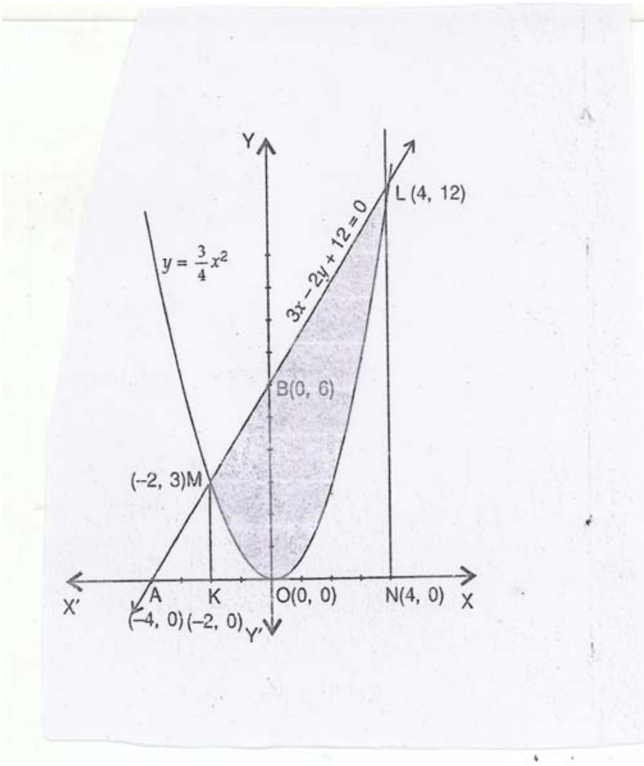


The graph of $\frac{x}{250}+\frac{3y}{500}$ <2.8 has no common point common with the feasible region
Min. value of Z = 2.80 and occurs when x= 400 and y= 200
Min. value of Z = RS 2.80 when x= 400 gm wheat and y= 200 gm rice .
(1m)

24. Find the area of the region included between the parabola $4y = 3x^2$ and the line 3x -2y + 12 = 0 .

ANS 24. Given parabola y = $\frac{3x^2}{4}$ (1)
and the line 3x -2y +12 = 0

The equation (1) represents a parabola with vertex at the origin O(0,0) and axis of the parabola is along the positive direction of y- axis as shown in the figure .
FIGURE is shown below



3x -2y +12 = 0

$\Rightarrow \frac{x}{-4} + \frac{y}{6} = 1$ (2)

Points of intersection of (1) and (2)

x=4 or x=-2
therefore y = 12 or y=3

Points are (4,12) and (-2,3) (2m)

Required area = (Area of trapezium LNKM i.e. area under the line 3x-2y +12 = 0 from -2 to 4) -(area under parabola y = $\frac{3x^2}{4}$ from -2 to 4)

$= \int_{-2}^{4} \left(\frac{3x+12}{2} \right) dx - \int_{-2}^{4} \frac{3x^2}{4} dx$ (2m)

$= \left[\frac{3}{4}x^2 + 6x - \frac{x^3}{4} \right]_{-2}^{4}$

=12+24-16-3+12-2

27 sq. units ANS (2m)

25. Evaluate : $\int_1^3 (2x^2 + 3x + 7)$ dx as limit of sums

ANS Let f(x) = $(2x^2 + 3x + 7)$

a=1 , b= 3 h = 2/n therefore nh = 2 (1m)

$\int_1^3 (2x^2 + 3x + 7)$ dx =

$\lim_{h \to 0}$ h(f(1) +f(1+h) +f(1+2h) +-------+f(1+(n-1)h) (1m)

=$\lim_{h \to 0}$ h (2 $(1)^2$+3 (1) +7) + $(2(1+h)^2$ +3(1+h)+7) +-----------$(2(1+(n-1)h)^2$ +3(1+(n-1)h+7)

$\lim_{h \to 0}$ h (12 +12 +-------n terms) +2h^2 (1^2+2^2+----------$(n-1)^2$) +7h(1+2+-----(n-1)) (1m)

$\lim_{h \to 0}$ h (12n +2h^2 $\frac{n(n-1)(2n-1)}{6} + 7h\frac{n(n-1)}{2}$) (1m)

= $\lim_{h \to 0}$ h (12(nh) + $\frac{1}{3}(nh-n)(nh)(2nh-h) + \frac{7}{2}(nh)(nh-h)$)

=24+ $\frac{1}{3} \times 2 \times 2 \times 4 + \frac{7}{2} \times 2 \times 2$ = **130/3 ANS** (2m)

26. Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards . Find the mean , variance and standard deviation of the number of kings .

ANS Let X denote the number of kings in a draw of two cards . X is random variable which can assume the values 0,1,or 2.

P(X= 0) =P(no king)= $\frac{{}^{48}C_2}{{}^{52}C_2} = \frac{188}{221}$ (1m)

P(X= 1) = P (one king and one non- king)= $\frac{{}^{4}C_1 \times {}^{48}C_1}{{}^{52}C_2} = \frac{32}{221}$ (1m)

P(X= 2) P (two kings) = $\frac{^{4}C_2}{^{52}C_2} = \frac{1}{221}$ (1m)

The probability distribution of X is

X	0	1	2
P(X)	188/221	32/221	1/221

Mean of X = E(X)= $\sum_{i=1}^{n} x_i\, p(x_i)$ = 34/221 (1m)

$E(X^2) = 36/221$
$Var(X) = E(X^2) - (E(X))^2 = 6800/(221)^2$ (1m)

$\sigma_x = \sqrt{VAr(X)} = \frac{\sqrt{6800}}{221} = 0.37$ ANS (1m)

27. A right circular cone is circumscribed about a right circular cylinder of radius r and altitude h. Show that the volume of the cone is least , when
    i) the altitude of the cone is equal to 3 times the altitude of the cylinder .
    ii) the radius of the cone is equal to 3/2 times the radius of the cylinder .

Ans Let x and y be the radius and altitude of the height of the right circular cone which is circumscribed about the right circular cylinder of radius r and height h .

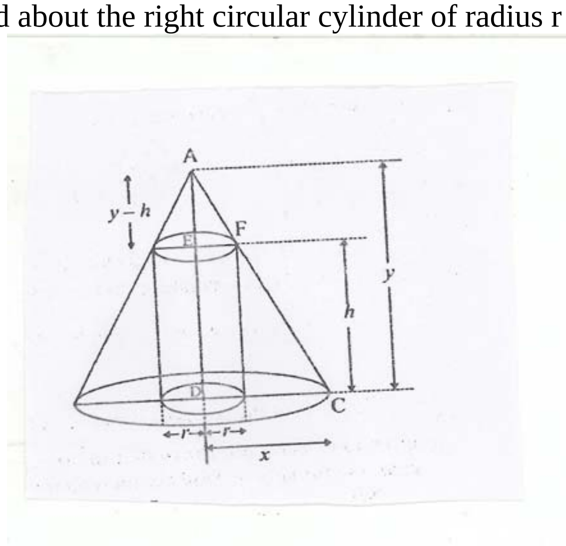


FIGURE (1/2 m)

From the fig ED = h , EF = r , DC = x , AD = y , AE = y-h .
From similar triangles AEF and ADC , we have

$$\frac{AE}{AD} = \frac{EF}{DC} \qquad \text{(1m)}$$

$$\frac{y-h}{y} = \frac{r}{x}$$

y = $\frac{hx}{x-r}$

If V denote the volume of a right circular cone , then

V= $\frac{1}{3}\pi x^2 y \quad = \frac{\pi h}{3}\left(\frac{x^3}{x-r}\right)$

$$\frac{dV}{dx} = \frac{\pi h}{3}\left(2x + r - \frac{r^3}{(x-r)^2}\right) \qquad \text{(1m)}$$

$$\frac{d^2V}{dx^2} = \frac{\pi h}{3}\left(2 - \frac{2r^3}{(x-r)^3}\right)$$

Setting dV /dx = 0

x = $\frac{3r}{2}$ (1m)

Now $\left(\frac{d^2V}{dx^2}\right)_{x=3r/2} = \frac{\pi h}{3}\left(2 - \frac{2r^3}{(x-r)^3}\right) = \frac{\pi h}{3} > 0$ (1m)

Thus the volume of a right circular cone is least when x= 3r/2 .

When x= 3r/2 then y = 3h (1m)

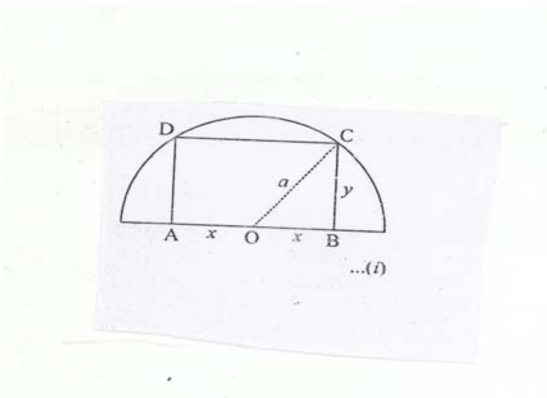
Hence the volume of the right circular cone is least when
i) y = 3h i.e. altitude of the right circular cone = 3(altitude of the cylinder)
ii) x=3r/2 i.e. the radius of the cone is equal to 3/2 times the radius of the cylinder .
(1/2m)
Hence proved .

OR

A rectangle is inscribed in a semi- circle of radius 'a' with one of its sides on the diameter of semi- circle . Find the dimensions of the rectangle so that its area is maximum . Find the area also .

Ans Let ABCD be the rectangle of length 2x and breadth y inscribed in a semi circle of radius a with centre O.

FIGURE (1/2 m)



In right triangle OBC we have $OC^2 = OB^2 + BC^2$

$y^2 = a^2 - x^2$ (1)

A = 2xy

$A^2 = 4x^2y^2$

$Z = 4x^2y^2$ (let $Z = A^2$) (1m)

$Z = 4(a^2x^2 - x^4)$

Diff. w.r.t x we get

$$\frac{d^2Z}{dx^2} = 4(2a^2 - 12x^3)$$

dZ/dx= 0

gives $x = \frac{a}{\sqrt{2}}$ (1m)

Second order derivative at $x = \frac{a}{\sqrt{2}}$ $\left.\frac{d^2Z}{dx^2}\right|_{x=a/\sqrt{2}} = 4(2a^2 - 12x^3)$

$= -16a^2 < 0$ (ive) (1m)

Therefore Z is maximum when $x=\frac{a}{\sqrt{2}}$

From (1) $y = \frac{a}{\sqrt{2}}$

Length of the rectangle $2x = 2\frac{a}{\sqrt{2}} = \sqrt{2}a$

Breadth of the rectangle = $y = \frac{a}{\sqrt{2}}$ (1m)

Z is maximum when the length of the rectangle is $\sqrt{2}a$ and breadth of the rectangle is $\frac{a}{\sqrt{2}}$

$\Rightarrow$ A is maximum when the length of the rectangle is $\sqrt{2}a$ and breadth of the rectangle is $\frac{a}{\sqrt{2}}$

Therefore max. area of the rectangle = $(2x)\,y = \sqrt{2}a\,\frac{a}{\sqrt{2}} = a^2$ sq.units . (2m)

28.Using elementary transformations, find the inverse of the matrix .

$$\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$$

ANS $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Applying $R_2 \rightarrow R_2 + 3R_1$ and $R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

$R_1 \rightarrow R_1 + 3R_3$ and $R_2 \rightarrow R_2 + 9R_3$

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 0 & 25 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -15 & 1 & 9 \\ -2 & 0 & 1 \end{bmatrix} A \qquad \text{(2m)}$$

$R_3 \rightarrow -R_3$

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 0 & 25 \\ 0 & 1 & -4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -15 & 1 & 9 \\ 2 & 0 & -1 \end{bmatrix} A$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 2 & 0 & -1 \\ -15 & 1 & 9 \end{bmatrix} A \qquad \text{(2m)}$$

$R_3 \rightarrow R_3/25$

$$\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 2 & 0 & -1 \\ -3/5 & 1/25 & 9/25 \end{bmatrix} A$$

$R_1 \rightarrow R_1 - 10R_3$ and $R_2 \rightarrow +4R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2/25 & -3/5 \\ -2/5 & 4/25 & 11/25 \\ -3/5 & 1/25 & 9/25 \end{bmatrix} A$$

Hence $A^{-1} = \begin{bmatrix} 1 & -2/25 & -3/5 \\ -2/5 & 4/25 & 11/25 \\ -3/5 & 1/25 & 9/25 \end{bmatrix}$. (2m)

29. Find the equation of the plane passing through line of intersection of the planes $x - 2y + z = 1$ and $2x + y + z = 8$ and parallel to the ;line with direction ratios 1,2,1 .Find also the perpendicular distance of (1,1,1) from this plane .

ANS .29 The equation of a plane passing through the intersection the intersection of the given plane is

$(x - 2y + z - 1) + \lambda(2x + y + z - 8) = 0$

$(1+2\lambda)x + (\lambda-2)y + (\lambda+1)z - (1+8\lambda) = 0$ (1) (2m)

Let this plane be parallel to the line with direction ratios 1,2,1 . Then the normal to this plane is perpendicular to the line having direction ratios 1,2,1 .

$1.(1+2\lambda) + 2(\lambda-2) + 1(\lambda+1) = 0$

$\lambda = 2/5$. (1m)

Putting $\lambda = 2/5$.
In (1) we get

$(1+2\times2/5)x + (2/5-2)y + (2/5+1)z - (1+8\times2/5) = 0$

$9x - 8y + 7z - 21 = 0$. (1m)

This is the required equation of the plane .

Now , length of the perpendicular from (1,1,1) to $9x - 8y + 7z - 21 = 0$.
is equal to

$$= \frac{|9\times1 - 8\times1 + 7\times1 - 21|}{\sqrt{9^2 + (-8)^2 + (7)^2}} = \frac{13}{\sqrt{194}} \text{ units .}$$ (2m)

END OF THE EXAMINTAION